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Quantum Mechanical Prediction of the Singlet State with Local Functions

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The following is a way of deriving the quantum mechanical (QM) prediction for the EPR-Bohm experiment [1] with local measurement functions. We have given in the Appendix a standard QM derivation for comparison. Consider an "entangled" pair of spin one-half particles, moving freely after production in opposite directions, with particles 1 and 2 subject, respectively, to spin measurements along independently chosen unit directions $\mathbf{a}$ and $\mathbf{b}$, which can be located at a spacelike distance from each other. If initially the pair has vanishing total spin, then the pair's quantum mechanical spin state would be the following "entangled" singlet state:

$$|\Psi_{\mathbf{n}}\rangle = \frac{1}{\sqrt{2\,V}}\Big\{|\mathbf{n},\,+\rangle_1 \otimes |\mathbf{n},\,-\rangle_2 \,-\, |\mathbf{n},\,-\rangle_1 \otimes |\mathbf{n},\,+\rangle_2\Big\}. \tag{1}$$

And,

$$\boldsymbol{\sigma}\cdot\mathbf{n}\,|\mathbf{n},\,\pm\rangle \,=\, \pm\,|\mathbf{n},\,\pm\rangle \tag{2}$$

describing the quantum mechanical eigenstates in which the particles have spin "up" or "down" in units of $\hbar = 2$, with $\boldsymbol{\sigma}$ being the familiar Pauli spin "vector" $(\sigma_x,\,\sigma_y,\,\sigma_z)$.

Quantum mechanically the rotational invariance of the singlet state $|\Psi_{\mathbf{n}}\rangle$ ensures that the expectation values of the individual spin observables $\boldsymbol{\sigma}_1\cdot\mathbf{a}$ and $\boldsymbol{\sigma}_2\cdot\mathbf{b}$ are

$$\begin{aligned} \mathcal{E}_{q.m.}(\mathbf{a}) \,&=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\mathbb{1}\,|\Psi_{\mathbf{n}}\rangle \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0 \\ \text{and}\quad \mathcal{E}_{q.m.}(\mathbf{b}) \,&=\, \langle\Psi_{\mathbf{n}}|\,\mathbb{1}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0\,, \end{aligned} \tag{3}$$

where $\mathbb{1}$ is the identity matrix and the results are the same for the left handed state.

We will also use the well known identities,

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\,(\boldsymbol{\sigma}\cdot\mathbf{b}) \,=\, \mathbf{a}\cdot\mathbf{b}\,\mathbb{1} \,+\, i\,\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}), \tag{4a}$$

$$(\boldsymbol{\sigma}\cdot\mathbf{b})\,(\boldsymbol{\sigma}\cdot\mathbf{a}) \,=\, \mathbf{a}\cdot\mathbf{b}\,\mathbb{1} \,-\, i\,\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}), \tag{4b}$$

which follows from the non-commutativity of products of the Pauli matrices σ_j $(j = x, y, z)$ defined by the algebra

$$\sigma_j\sigma_k \,=\, \delta_{jk}\,\mathbb{1} \,+\, i\,\epsilon_{jkl}\,\sigma_l\,, \tag{5}$$

where δ_{jk} is the Kronecker delta, $i \equiv \sqrt{-1}$ is the unit imaginary, and ϵ_{jkl} is the Levi-Civita alternating symbol.

It is now possible to construct some manifestly local measurement functions that agree with the eigenvalues of the observable operators which involve spins being detected by detectors:

$$A(\mathbf{a},\,\mathbf{s}) := -\lim_{\mathbf{s}\to\,\mathrm{sgn}(\mathbf{a}\cdot\mathbf{s})\mathbf{a}}\Big[\langle\phi_{\mathbf{n}}|\{(\boldsymbol{\sigma}\cdot\mathbf{a})\,(\boldsymbol{\sigma}\cdot\mathbf{s})\}|\phi_{\mathbf{n}}\rangle\Big] = \mp 1 \tag{6}$$

$$\text{and}\quad B(\mathbf{b},\,\mathbf{s}) := +\lim_{\mathbf{s}\to\,\mathrm{sgn}(\mathbf{b}\cdot\mathbf{s})\mathbf{b}}\Big[\langle\chi_{\mathbf{n}}|\{(\boldsymbol{\sigma}\cdot\mathbf{s})\,(\boldsymbol{\sigma}\cdot\mathbf{b})\}|\chi_{\mathbf{n}}\rangle\Big] = \pm 1, \tag{7}$$

where

$$|\phi_{\mathbf{n}}\rangle = \frac{1}{\sqrt{2}}\Big\{|\mathbf{n},\,+\rangle_1|\mathbf{n},\,-\rangle_3\Big\} \tag{8}$$

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$$\text{and}\quad |\chi_{\mathbf{n}}\rangle = \frac{1}{\sqrt{2}}\Big\{|\mathbf{n},\,+\rangle_4|\mathbf{n},\,-\rangle_2\Big\}. \tag{9}$$

Here $\boldsymbol{\sigma}\cdot\mathbf{a}$ and $\boldsymbol{\sigma}\cdot\mathbf{b}$ represent the detectors of Alice and Bob with no angular momentum at time of detection, and $\boldsymbol{\sigma}\cdot\mathbf{s}$ represents the spin of the fermions they receive. The limits express the action of the polarizers at the detection stations. Note here that $|\phi_{\mathbf{n}}\rangle$ and $|\chi_{\mathbf{n}}\rangle$ are simple products and are now different particles. And that the original singlet is now split between two different simple product bra-kets.

Note that these measurement functions represent simultaneous detection processes occurring at two possibly space-like separated observation stations of Alice and Bob. Although occurring simultaneously, $A(\mathbf{a},\,\lambda)$ and $B(\mathbf{b},\,\lambda)$ are independent physical processes that are *not* subject to the conservation of the initial zero spin angular momentum. Before proceeding with the product calculation, we will implement a notation simplification, $\mu_{\mathbf{n}} = \mathrm{sgn}(\mathbf{n}\cdot\mathbf{s})\mathbf{n}$. Upon using the "product of limits equal to limits of product" rule, leads to the expectation value calculated as follows:

$$\begin{aligned}
\mathcal{E}_{L.R.}(\mathbf{a},\,\mathbf{b}) &= \lim_{n>>1}\left[\frac{1}{n}\sum_{k=1}^{n} A(\mathbf{a},\,\mathbf{s}^k)\otimes B(\mathbf{b},\,\mathbf{s}^k)\right] &(10)\\
&= \lim_{n>>1}\left[\frac{1}{n}\sum_{k=1}^{n}\left[-\lim_{\mathbf{s}\to\,\mu_{\mathbf{a}}}\langle\phi_{\mathbf{n}}|\{(\boldsymbol{\sigma}\cdot\mathbf{a})\,(\boldsymbol{\sigma}\cdot\mathbf{s})\}|\phi_{\mathbf{n}}\rangle\right]\otimes\left[\lim_{\mathbf{s}\to\,\mu_{\mathbf{b}}}\langle\chi_{\mathbf{n}}|\{(\boldsymbol{\sigma}\cdot\mathbf{s})\,(\boldsymbol{\sigma}\cdot\mathbf{b})\}|\chi_{\mathbf{n}}\rangle\right]\right. &(11)\\
&= -\left[\lim_{\mathbf{s}\to\,\mu_{\mathbf{a}}}\{(\mathbf{a}\cdot\mathbf{s})\}\right]\left[\lim_{\mathbf{s}\to\,\mu_{\mathbf{b}}}\{(\mathbf{s}\cdot\mathbf{b})\}\right] &(12)\\
&= -\left[\lim_{\mathbf{s}\to\,\mu_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{s})(\mathbf{s}\cdot\mathbf{b})\right] &(13)\\
&= -\,\mathbf{a}\cdot\mathbf{b}\,. &(14)
\end{aligned}$$

Thus we obtain the correct result via a completely local process.

Appendix A: Standard QM Derivation

Considered a pair of spin one-half particles, moving freely after production in opposite directions, with particles 1 and 2 subject, respectively, to spin measurements along independently chosen unit directions $\mathbf{a}$ and $\mathbf{b}$, which can be located at a spacelike distance from each other. If initially the pair has vanishing total spin, then its quantum mechanical (QM) spin state would be the entangled singlet state

$$|\Psi_{\mathbf{n}}\rangle = \frac{1}{\sqrt{2}}\Big\{|\mathbf{n},\,+\rangle_1\otimes|\mathbf{n},\,-\rangle_2\,-\,|\mathbf{n},\,-\rangle_1\otimes|\mathbf{n},\,+\rangle_2\Big\}, \tag{A1}$$

with $\mathbf{n}$ indicating an arbitrary unit direction, and

$$\boldsymbol{\sigma}\cdot\mathbf{n}\,|\mathbf{n},\,\pm\rangle \,=\, \pm\,|\mathbf{n},\,\pm\rangle \tag{A2}$$

describing the quantum mechanical eigenstates in which the particles have spin "up" or "down" in units of $\hbar = 2$. Here $\boldsymbol{\sigma}$ is the familiar Pauli spin "vector" $(\sigma_x,\,\sigma_y,\,\sigma_z)$. Our interest lies in comparing the quantum predictions of spin correlations between the two remote subsystems with those derived from any locally causal theory.

Now, quantum mechanically the rotational invariance of the state $|\Psi_{\mathbf{n}}\rangle$ ensures that the expectation values of the individual spin observables $\boldsymbol{\sigma}_1\cdot\mathbf{a}$ and $\boldsymbol{\sigma}_2\cdot\mathbf{b}$ are

$$\begin{aligned}
\mathcal{E}_{q.m.}(\mathbf{a}) &= \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\mathbb{1}\,|\Psi_{\mathbf{n}}\rangle \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0\\
\text{and}\quad \mathcal{E}_{q.m.}(\mathbf{b}) &= \langle\Psi_{\mathbf{n}}|\,\mathbb{1}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0\,,
\end{aligned} \tag{A3}$$

where $\mathbb{1}$ is the identity matrix. The expectation value of the joint observable $\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}$, on the other hand, is

$$\mathcal{E}_{q.m.}(\mathbf{a},\,\mathbf{b}) \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\,\otimes\,\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,, \tag{A4}$$

regardless of the relative distance between the two remote locations represented by the unit vectors $\mathbf{a}$ and $\mathbf{b}$. The last result can be derived using the following calculation for the singlet state,

$$\mathcal{E}_{q.m.}(\mathbf{a},\,\mathbf{b}) \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}\cdot\mathbf{a}\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \tag{A5}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big[(1\ \ 0)\,(\boldsymbol{\sigma}\cdot\mathbf{a})\begin{pmatrix}1\\0\end{pmatrix}(0\ \ 1)\,(\boldsymbol{\sigma}\cdot\mathbf{b})\begin{pmatrix}0\\1\end{pmatrix} - (0\ \ 1)\,(\boldsymbol{\sigma}\cdot\mathbf{a})\begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0)\,(\boldsymbol{\sigma}\cdot\mathbf{b})\begin{pmatrix}0\\1\end{pmatrix} \\
&\quad - (1\ \ 0)\,(\boldsymbol{\sigma}\cdot\mathbf{a})\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1)\,(\boldsymbol{\sigma}\cdot\mathbf{b})\begin{pmatrix}1\\0\end{pmatrix} + (0\ \ 1)\,(\boldsymbol{\sigma}\cdot\mathbf{a})\begin{pmatrix}0\\1\end{pmatrix}(1\ \ 0)\,(\boldsymbol{\sigma}\cdot\mathbf{b})\begin{pmatrix}1\\0\end{pmatrix}\Big], \qquad \text{(A6)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Big[(1\ \ 0)\begin{pmatrix}a_z & a_x - ia_y\\ a_x + ia_y & -a_z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}(0\ \ 1)\begin{pmatrix}b_z & b_x - ib_y\\ b_x + ib_y & -b_z\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \\
&\quad - (0\ \ 1)\begin{pmatrix}a_z & a_x - ia_y\\ a_x + ia_y & -a_z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0)\begin{pmatrix}b_z & b_x - ib_y\\ b_x + ib_y & -b_z\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \\
&\quad - (1\ \ 0)\begin{pmatrix}a_z & a_x - ia_y\\ a_x + ia_y & -a_z\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1)\begin{pmatrix}b_z & b_x - ib_y\\ b_x + ib_y & -b_z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\
&\quad + (0\ \ 1)\begin{pmatrix}a_z & a_x - ia_y\\ a_x + ia_y & -a_z\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}(1\ \ 0)\begin{pmatrix}b_z & b_x - ib_y\\ b_x + ib_y & -b_z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\Big], \qquad \text{(A7)}
\end{aligned}
$$

$$
= \frac{1}{2}\{(-a_z b_z) + (-a_x b_x - a_y b_y) + (-a_x b_x - a_y b_y) + (-a_z b_z)\}, \qquad \text{(A8)}
$$

$$
= -\,\mathbf{a}\cdot\mathbf{b}\,. \qquad \text{(A9)}
$$

[1] D. Bohm and Y. Aharonov, *Discussions of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky*, Phys. Rev., **108**, 1079 (1957).